Expanding Universes

Cosmology Block Course 2013

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Contents

Preparation for cosmological model-building

One key property of the universe:

- Homogeneous and isotropic on large scales (>100 Mpc)
- Average density rather small

Make this the first axiom of cosmological model-building:

Cosmological principle

On large scales, on average, the universe is homogeneous and isotropic

Cosmological model-building: strategy

Two-step model-building strategy guided by the cosmological principle:

1. Build idealized exactly homogeneous and isotropic models: **Friedmann-Lemaître-Robertson-Walker, FLRW** (exact family of solutions; this lecture)
2. Add inhomogeneities on smaller scales as perturbations (BMS's lecture)

Homogeneous and isotropic universes

Naïvely: A homogeneous universe is the same everywhere (in particular: density ρ =const.).

But: general relativity is a covariant theory — all coordinate systems admissible!

Relativistic definition: There exists a coordinate choice so that, at each fixed coordinate time, space is homogeneous (foliation).

(More rigorous definition: $\Rightarrow$ isometries and Killing vectors, way beyond scope of this course.)

Choice of time coordinate

Assume there is a *cosmic substrate* — matter (think: galaxy-size dust particles) that, for a given choice of time and space coordinates, is at rest and evenly distributed.

Cosmic time: time coordinate for which there can be explicit isotropy and homogeneity (this fixes simultaneity); time differences at the same point in space are proper time differences for substrate particles.

In consequence:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (\text{spatial part of metric}).$$

Choice of spatial metric: Euclidean

Rigorous route: Killing vectors & form invariance, cf. sec. 13 in Weinberg (1972)

Simpler question: What can we think of?

Euclidean space:

$$ds^2 = dx^2 + dy^2 + dz^2 \equiv d\vec{x}^2.$$

$$ds^2 = (dx, dy, dz) \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = d\vec{x}^T \cdot d\vec{x}$$

...this is invariant under translations, since $d(\vec{x} + \vec{a}) = d\vec{x}$ and under rotation, since $\vec{x} \mapsto M\vec{x}$ with $M \in SO(3)$ means

$$d(M\vec{x})^T \cdot d(M\vec{x}) = d\vec{x}^T \cdot M^T \cdot M \cdot d\vec{x} = d\vec{x}^T \cdot d\vec{x}.$$

Choice of spatial metric: Spherical

What other homogeneous, isotropic spaces are there?

Think spherical; a spherical surface S^{n-1} embedded in $\mathbb{R}^n$ is defined as the union of all points with n-dimensional coordinates x_i where

$$\sum_{i=1}^{n} x_i^2 = R^2$$

with R the radius of the sphere. Two-sphere S^2: ordinary spherical surface in space.

At least locally: Use $n-1$ of the coordinates as coordinates on the surface, $\vec{x}$; one coordinate as embedding coordinate, ξ, then

$$\mathrm{d}s^2 = \mathrm{d}\vec{x}^2 + \mathrm{d}\xi^2 \quad \text{where} \quad \xi^2 + \vec{x}^2 = R^2.$$

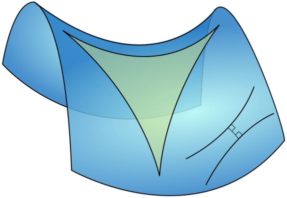
Choice of spatial metric: Spherical

$$\mathrm{d}s^2 = \mathrm{d}\vec{x}^2 + \mathrm{d}\xi^2 \quad \text{where} \quad \xi^2 + \vec{x}^2 = R^2$$

is invariant under rotations $M \in SO(4)$, which include homogeneity (any point can be rotated into any other point) and isotropy (any tangent vector can be rotated in any direction).

Easiest to see for $S^2 \in \mathbb{R}^3$: For each point P, one rotation (through embedding centerpoint and P) that will rotate space around P (isotropy), and two rotations that will shift the point into any given other point (homogeneity).

Choice of spatial metric: Hyperbolical



$$ds^2 = d\vec{x}^2 - d\xi^2 \quad \text{where} \quad \xi^2 - \vec{x}^2 = R^2.$$

Higher-dimensional analogue of a saddle; invariant under $R \in SO(3,1)$.

This is the Lorentz group: SO(3) rotations (isotropy around each given point) and 3 Lorentz boosts that take the point into an arbitrary other point (homogeneity).

Unifying the spherical and hyperbolical spaces

Rescale $\vec{x} \mapsto \vec{x}/R$ and $\xi \mapsto \xi/R$:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\vec{x}^2 \pm \mathrm{d}\xi^2\right] \quad \text{where} \quad \xi^2 \pm \vec{x}^2 = 1.$$

From the constraint equation,

$$\mathrm{d}(\xi^2 \pm \vec{x}^2) = 0 = 2(\xi \mathrm{d}\xi \pm \vec{x} \cdot \mathrm{d}x)$$

relates the differentials. Substitute in metric to get unconstrained version:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\vec{x}^2 \pm \frac{(\vec{x} \cdot \mathrm{d}\vec{x})^2}{1 \mp \vec{x}^2}.\right]$$

Unifying the spherical and hyperbolical spaces

Introduce parameter $K = +1, 0, -1$ to write all three metrics in the same form:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\vec{x}^2 + K \frac{(\vec{x} \cdot \mathrm{d}\vec{x})^2}{1 - K\vec{x}^2} \right]$$

where

$$K = \begin{cases} +1 & \text{spherical space} \\ \;\;0 & \text{Euclidean space} \\ -1 & \text{hyperbolical space} \end{cases}$$

Spherical coordinates

Introduce spherical coordinates r, θ, ϕ via

$$\begin{aligned} x &= r \cdot \sin\theta \cdot \cos\phi \\ y &= r \cdot \sin\theta \cdot \sin\phi \\ z &= r \cdot \cos\theta \end{aligned}$$

(in the usual way — think about θ as latitude and ϕ as longitude).
Direct calculation shows:

$$\mathrm{d}\vec{x}^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \equiv \mathrm{d}r^2 + r^2\mathrm{d}\Omega.$$

Also, $\vec{x}^2 = r^2$ and $\vec{x} \cdot \mathrm{d}x = r\mathrm{d}r$.

Spherical coordinates

Re-write the metric accordingly:

$$\mathrm{d}s^2 = R^2 \left(\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega \right).$$

This is nice and simple!

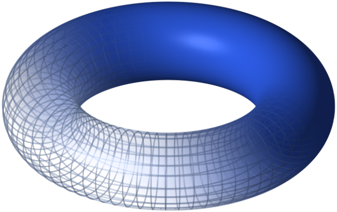
A caveat: global vs. local

The metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2\mathrm{d}\Omega.$$

describes space *locally*.

Globally, there is *topology* to consider — e.g. a flat metric can belong to infinite Euclidean space, but also, say, to a torus (a patch of Euclidean space with certain identifications).



⇒ Later on, we will learn of a possibility how a finite universe might be identified (cosmic background radiation)

A caveat: global vs. local

- $K = 0$: 18 topologically different forms of space. Some inifinite, some finite.
- $K = +1$: inifinitely many topologically different forms. All are finite.
- $K = -1$: infinitely many topologically different forms of space. Some inifinite, some finite.

Back to spacetime

R can only depend on t (homogeneity): $R \to a(t)$.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\mathrm{d}\vec{x}^2 + K \frac{(\vec{x} \cdot \mathrm{d}\vec{x})^2}{1 - K\vec{x}^2} \right] = -\mathrm{d}t^2 + a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega \right]$$

$a(t)$ is the **cosmic scale factor**

This is the **Friedmann-Robertson-Walker-Metric** — unique description for homogeneous and isotropic spaces.

Properties of the FRW metric

Note: We haven't invoked Einstein's equations yet! What we derive now follows from symmetry!

$$ds^2 = dt^2 + a(t)^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right] = -dt^2 + a(t)^2 \tilde{g}(\vec{x})_{ij} dx^i\, dx^j$$

means that for any two spatial vectors $v^\mu = (0, \vec{u}), w^\mu = (0, \vec{w})$ we have

$$g(v, w) = a(t)^2\ \tilde{g}(\vec{v}, \vec{w})$$

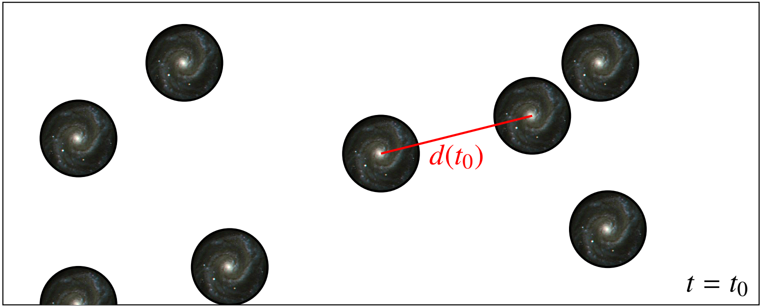
⇒ distance ratios and angles preserved over time!

The role of the scale factor

The role of the scale factor

Distances between galaxies

Consider galaxies in the Hubble flow:



All distances change as $d(t) = \dfrac{a(t)}{a(t_0)} \cdot d(t_0)$.

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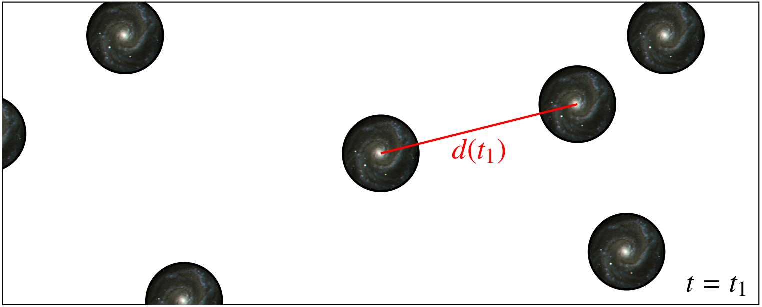

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Taylor expansion of the scale factor

Generic Taylor expansion:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\ddot{a} \cdot (t - t_0)^2 + \ldots$$

Re-define the expansion parameters by introducing two functions

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} \quad \text{and} \quad q(t) \equiv -\frac{\ddot{a}(t)a(t)}{\dot{a}(t)^2}$$

and corresponding constants

$$H_0 \equiv H(t_0) \quad \text{and} \quad q_0 \equiv q(t_0)$$

$$a(t) = a_0\left[1 + (t - t_0)H_0 - \frac{1}{2}q_0 H_0^2 (t - t_0)^2 + \ldots\right]$$

Some nomenclature and values 1/2

t_0 is the standard symbol for the **present time**. If coordinates are chosen so cosmic time $t = 0$ denotes the time of the big bang (phase), then t_0 is the **age of the universe**. Sometimes, the age of the universe is denoted by τ.

$H(t)$ is the **Hubble parameter** (sometimes misleadingly *Hubble constant*)

$H_0 \equiv H(t_0)$ is the **Hubble constant**. Current values are (cf. later lecture) around

$$H_0 = 70\,\frac{\mathrm{km/s}}{\mathrm{Mpc}}.$$

Some nomenclature and values 2/2

Sometimes, the Hubble constant is written as

$$H_0 = h \cdot 100\ \frac{\mathrm{km/s}}{\mathrm{Mpc}}$$

to keep ones options open with h the **dimensionless Hubble constant**.

The inverse of the Hubble constant is the **Hubble time** (cf. the linear case and the models later on).

$$\frac{1}{h \cdot 100 \frac{\mathrm{km/s}}{\mathrm{Mpc}}} \approx h^{-1} \cdot 10^{10}\ \mathrm{a}.$$

Matter at rest in an FRW universe

Our assumption: Floating substrate of particles. Is this consistent? Can particles be at rest?

We need the geodetic equations to tell us:

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\,\dot{x}^{\nu}\,\dot{x}^{\rho} = 0$$

Particle at rest has four-velocity:

$$\dot{x}^0 = 1 \quad \text{and} \quad \dot{x}^i = 0.$$

Does this solve the geodetic equation? ⇒ **Exercise**

Light in an FRW universe

For light, often easier to use $\mathrm{d}s^2 = 0$ instead of the geodetic equation.

Also: use symmetries! Move origin of your coordinate system wherever convenient. Look only at radial movement.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - Kr^2} + r^2 \mathrm{d}\Omega \right]$$

becomes

$$\mathrm{d}t = \pm \frac{a(t)\,\mathrm{d}r}{\sqrt{1 - Kr^2}}.$$

Light in an FRW universe

Integrate to obtain

$$\int_{t_2}^{t_1} \frac{\mathrm{d}t}{a(t)} = \pm \int_{r_2}^{r_1} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

Plus/minus: light moving towards us or away from us.

The key to astronomical observations in an FRW universe:

$$\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_{0}^{r_1} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

where, by convention, t_0 is present time, $t_1 < t_0$ emission time of particle, r_1 (constant) coordinate value for distant source.

Light signals chasing each other 1/2

Imagine two signals leaving a distant galaxy at $r = r_1$ at consecutive times t_1 and $t_1 + \delta t_1$, arriving at t_0 and δt_0. Then

$$\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}.$$

and

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^{r_1} \frac{\mathrm{d}r}{\sqrt{1-Kr^2}}$$

$$\int_{t_0}^{t_0+\delta t_0} \frac{\mathrm{d}t}{a(t)} - \int_{t_1}^{t_1+\delta t_1} \frac{\mathrm{d}t}{a(t)} = 0.$$

Light signals chasing each other 2/2

For small δt,

$$\int_{\bar{t}}^{\bar{t}+\delta t} f(t)\,\mathrm{d}t \approx f(\bar{t}) \cdot \delta t,$$

so in our case

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

Signals could be anything — in particular: consecutive crests (or troughs) of light waves, $f \propto 1/\delta t$:

$$\frac{f_0}{f_1} = \frac{a(t_1)}{a(t_0)},\text{ wavelengths change as } \frac{\lambda_0}{\lambda_1} = \frac{a(t_0)}{a(t_1)}.$$

Frequency shift by expansion

Redshift defined as

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1$$

or

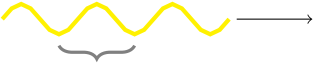
$$1 + z = \frac{a(t_0)}{a(t_1)}$$

For co-moving galaxies: z is directly related to r_1. For monotonous $a(t)$: distance measure.

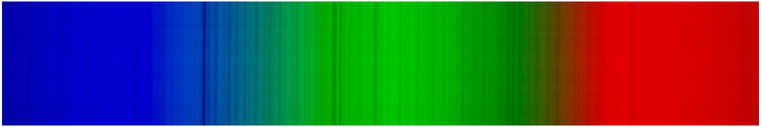
Relation depends on dynamics ⇒ later!

Cosmological redshift

Wavelength scaling with scale factor:



Rotverschiebung für $a(t_0) > a(t_1)$; Blauverschiebung $a(t_0) < a(t_1)$



For “nearby” galaxies. . .

. . . use the Taylor expansion $a(t) = a(t_0)[1 + H_0(t - t_0) + O((t - t_0)^2)]$:

$$1 - z \approx \frac{1}{1+z} = \frac{a(t_1)}{a(t_0)} \approx 1 + H_0(t_1 - t_0)$$

or

$$z \approx H_0(t_0 - t_1) \approx H_0 d$$

for small z, small $t_0 - t_1$.

This is Hubble’s law.
Originally discovered by Alexander Friedmann (cf. Stigler’s law).

Pedestrian derivation of Hubble's law and redshift

For scale factor expansion, $d(t) = a(t)/a(t_0) \cdot d(t_0)$:
"Instantaneous speed" of a galaxy

$$v(t) = \frac{\dot{a}(t)}{a(t)} d(t) = H(t)\, d(t) \approx H_0\, d(t).$$

Classical (moving-source) Doppler effect:

$$z = v$$

in other words:

$$z = H_0\, d.$$

Moving galaxies?

. . . so are galaxies really moving with $v = H_0\, d$?

Exact form for redshift

$$1 + z = \frac{a(t_0)}{a(t_1)}$$

shows that it's not about motion – it's about what happens to the light on its way!

Would $v > c$ be a problem? (For $H_0 = (70 km/s)/Mpc)$ from 4.3 Gpc onwards.)

Remember the equivalence principle: SR does not care about global GR effects, as long as locally all is well.

Changing redshifts

z depends on the observing time, as well! For one and the same object (r_1 fixed), and evaluated at t_0:

$$\frac{\mathrm{d}z}{\mathrm{d}t_0} = \frac{\mathrm{d}}{\mathrm{d}t_0}\left[\frac{a(t_0)}{a(t_1)}\right] = \frac{\dot{a}(t_0)}{a(t_1)} - \frac{a(t_0)\dot{a}(t_1)}{a(t_1)^2}\frac{\mathrm{d}t_1}{\mathrm{d}t_0}.$$

But

$$\frac{\mathrm{d}t_1}{\mathrm{d}t_0} = \frac{1}{1+z}$$

(that was the redshift argument). Insert Hubble function:

$$H(t_1) = H_0(1+z) - \frac{\mathrm{d}z}{\mathrm{d}t_0}.$$

Measure the change in z, and you can reconstruct the past!

The evolution of the scale factor

Up to now, all our conclusions drawn from metric — derived by *symmetry*.

To find the explicit form of $a(t)$, we need Einstein's equations,

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

— in particular: we need to assume a (homogeneous, isotropic...) stress-energy tensor $T_{\mu\nu}$. Choose the perfect fluid,

$$T^{\mu\nu} = (\rho + p)\, u^{\mu} u^{\nu} + p g^{\mu\nu}$$

(where we have generalized $\eta \to g$), then implement isotropy: $u^{\mu} = (1, \vec{0})$ — the substrate (gas, . . .) is, on average, at rest in the cosmic reference frame.

Solving Einstein's equations for FRW

00 component of Einstein's eq.:

$$3\,\frac{\dot{a}^2+K}{a^2}-\Lambda=8\pi G\,\rho$$

$i0$ components vanish. ij components give

$$2\frac{\ddot{a}}{a}+\frac{\dot{a}^2+K}{a^2}-\Lambda=-8\pi Gp$$

These are the **Friedmann equations**. Their solutions are the **Friedmann-Lemaître-Robertson-Walker** (FLRW) universes.

(And recall that $\rho_\Lambda=-p_\Lambda=\Lambda/8\pi G$.)

Re-casting the Friedmann equations

$$\frac{\dot{a}^2 + K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G\rho}{3}$$

and for $\dot{a} \neq 0$
(by differentiating the above and inserting the ij-equation)

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) = -3H(t)(\rho + p).$$

— this is nothing new: energy conservation for the ideal fluid stress-energy tensor in FRW spaces! ⇒ **Exercise**

The physics behind the Friedmann equations

Multiply

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

by a^3 and integrate:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\, a^3) + p\frac{\mathrm{d}a^3}{\mathrm{d}t} = 0.$$

The volume of a small ball $0 \leq r \leq r_1$ is

$$V = \iiint \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = a^3\; v(r_1).$$

Using this, rewrite

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V) + p\frac{\mathrm{d}V}{\mathrm{d}t} = 0.$$

The physics behind the Friedmann equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho V) + p\frac{\mathrm{d}V}{\mathrm{d}t} = 0.$$

but ρ is energy density — $\rho V = U$ is the system's energy!

$$\Rightarrow \quad \mathrm{d}U = -p\,\mathrm{d}V$$

— change in energy is the "expansion work".

If $p = 0$ (dust universe), $\mathrm{d}U = 0$, so mass is conserved!

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Physics behind the Friedmann: deceleration

Recombine Friedmann equations to give equation for $\ddot{a}$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

Almost Newtonian — but in general relativity, pressure is a source of gravity, as well! (E.g. stellar collapse.)

This leads to an expression for the deceleration parameter:

$$q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0)$$

(with ρ_0 and p_0 the present density/pressure).

Newtonian analogy

Using purely Newtonian reasoning, one can derive the Friedmann equations for dust for $K = 0$.

In that derivation, all the dust particles have started with initial velocities just right for scale-factor expansion to occur. Mass is conserved. The mutual gravitational attraction slows down the expansion.

Details ⇒ **Exercise**

Different equations of state

Now, assume equation of state $p = w\rho$. Then

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$

becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

which is readily integrated to

$$\rho \sim a^{-3(1+w)}.$$

This describes how the cosmic content is *diluted* by expansion.

How does density change with the scale factor?

Earlier on, we had looked a three different equations of state $p = w\rho$:

1. **Dust**: $w = 0 \quad \Rightarrow \quad \rho \sim 1/a^3$
2. **Radiation**: $w = 1/3 \quad \Rightarrow \quad \rho \sim 1/a^4$
3. **Scalar field/dark energy**: $w = -1 \quad \rho =$const.

Whenever these are the only important components, a universe can have different *phases* — depending on size, different components will dominate.

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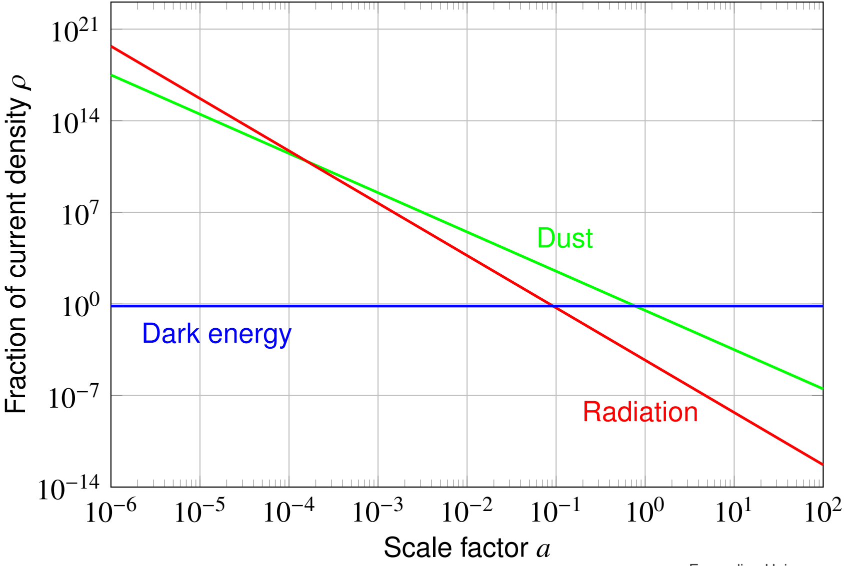
Different eras depending on the scale factor



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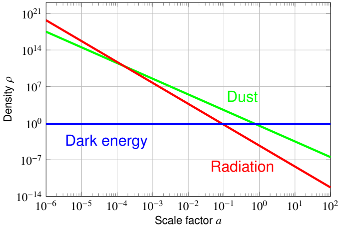


Two caveats:

- This says little about evolution — some values of a might not even be reached
- In reality, matter will change — particles might start as dust (non-relativistic) and, at smaller a, end up at high energies and thus as radiation (relativistic particles)

For small a: Radiation dominates!

$$\frac{\dot{a}^2 + K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G \rho}{3}$$

we can rewrite, using the scale-dependence of different components, as

$$\dot{a}^2 = -K + \frac{1}{3}\Lambda a^3 + \frac{8\pi G\, a_0^2}{3}\left[\rho_m\left(\frac{a_0}{a}\right) + \rho_R\left(\frac{a_0}{a}\right)^2\right]$$

with ρ_M (ρ_R) the matter (radiation) density at $t = t_0$.

As we go to smaller and smaller a (as in going into our own universe's past), curvature, Λ and matter (dust) become ever less important. Only radiation (including relativistic particles) matters.

For small a: Radiation dominates!

Rewrite Friedmann equation as

$$\dot{a}^2 = \left[\frac{8\pi G\rho_R a_0^2}{3}\right]\left(\frac{a_0}{a}\right)^2 .$$

Solve for $a(t)$ as

$$a(t) \propto \sqrt{t}$$

where $a(0) = 0$.

This will be the basis of all our models of the *early* universe.

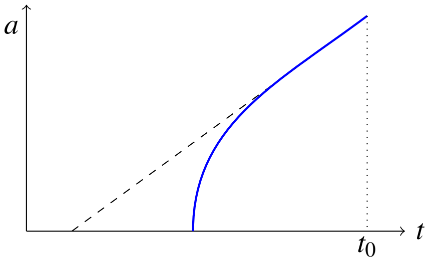
Convenient: Parameters decouple! Some important in the early universe, some only later!

The initial singularity

Combine $3\,\frac{\dot{a}^2+K}{a^2} - \Lambda = 8\pi G\rho$ and $2\frac{\ddot{a}}{a} + \frac{\dot{a}^2+K}{a^2} - \Lambda = -8\pi Gp$ to yield

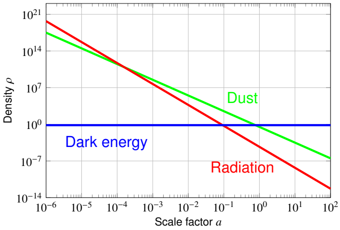
$$3\frac{\ddot{a}}{a} = -4\pi G(\rho + 3p) + \Lambda.$$

Shows that, for universes where Λ does not dominate, $\ddot{a}/a \leq 0$:



Initial singularity — special case of Hawking-Penrose theorems

If a universe becomes large, Λ dominates



Remember the deceleration parameter:

$$q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0).$$

Occasional misunderstanding: "Dark energy is negative, and acts like negative mass" – no: what accelerates the expansion is the negative pressure, $p_\Lambda = -\rho_\Lambda$. It dominates because of the factor 3!

Flatness problem 1/2

Once more

$$\frac{\dot{a}^2 + K}{a^2} = \frac{8\pi G\rho}{3}$$

(with Λ included in ρ). Define *time-dependent critical density*

$$\rho_c(t) = \frac{3H(t)^2}{8\pi G}$$

and re-write

$$\rho(t) - \rho_c(t) = \frac{3K}{8\pi G}\frac{1}{a(t)^2}$$

and with $\Omega(t) = \rho(t)/\rho_c(t)$ as

$$\left(1 - \frac{1}{\Omega(t)}\right)\rho(t)a^2 = \frac{3K}{8\pi G}.$$

Flatness problem 2/2

$$\left(1-\frac{1}{\Omega(t)}\right)\rho(t)a^2=\frac{3K}{8\pi G}.$$

if identically zero, $K=0$, then $\Omega(t)=1$.

But physics is rarely that exact (except if there's a mechanism for it). What if geometry is *close to Euclidean*, but not *exactly Euclidean*?

ρ increases faster than a^2 decreases as we go to earlier times $\Rightarrow$ deviation of $\Omega(t)$ must have been much smaller in the past than presently — finetuning problem known as *flatness problem*.

(see later: *inflationary models*)

Universes with dust and Λ

For the moment, let us concentrate on universes with negligible radiation (appropriate for the present state of our own universe).

Continuity equation shows that

$$\rho a^3 = \text{const.},$$

so the Friedmann equation for $\dot{a}^2$ becomes

$$(\dot{a})^2 = \frac{C}{a} + \frac{\Lambda a^2}{3} - K$$

where

$$C \equiv \frac{8\pi G\rho}{3} a^3 = \text{const.}.$$

The family of Friedmann solutions

$$(\dot{a})^2 = \frac{C}{a} + \frac{\Lambda a^2}{3} - K$$

- Eq. has a unique solution if we specify parameter values C, Λ, K, the initial value $a(t_0)$ at some time t_0, and the sign of $\dot{a}(t_0)$.
- Symmetries: $t \rightarrow -t$ and t-translations. We will focus on expanding solutions ($t \mapsto -t$ would then give a collapsing solution) and, where possible, choose $a = 0$ at $t = 0$.
- $a = 0$ is a singularity (eq. "blows up"). Hence, no solution will have regions of both positive and negative a. We restrict our analysis to positive a.

The trivial static solution

Trivially: $C = \Lambda = K = 0$ and $a =$const. is Minkowski spacetime.

Static solution

Other $a(t)$ =const. solutions? Problem: this means $\dot{a}(t) = 0$, so we need to get back to the original equations (setting $p = 0$ for dust)

$$3\,\frac{\dot{a}^2 + K}{a^2} - \Lambda = 8\pi G\rho \quad \text{and} \quad 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + K}{a^2} - \Lambda = 0.$$

These hold for $\dot{a}(t) = 0$ if

$$\frac{K}{a^2} = \Lambda = 4\pi G\rho.$$

Physical condition $\rho > 0$ means $K = +1$.

This is the **Einstein Universe** (Einstein 1917 — birth of the cosmological constant; static, finite in size). But: unstable!

Empty solutions

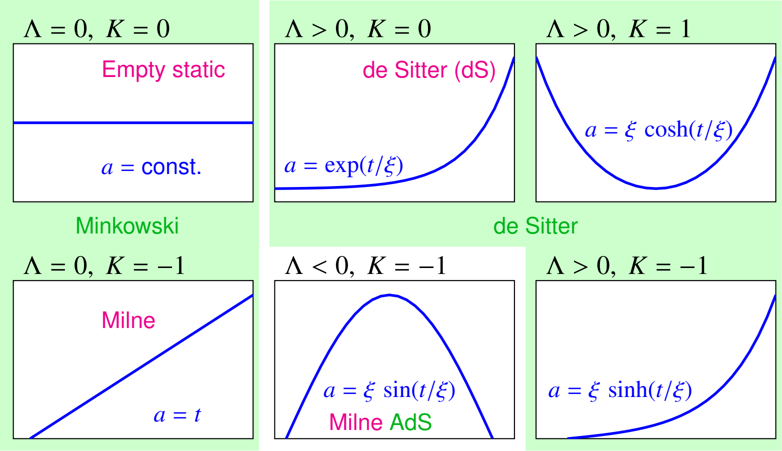
$\rho = 0$ or, alternatively "gravity switched off", $G = 0$: limiting cases or scalar-field universes.

Key equation (separation of variables):

$$\int \mathrm{d}t = \int \frac{\mathrm{d}a}{\sqrt{\frac{\Lambda a^2}{3} - K}}$$

Rescaling a, some cases (where the square root is real) can readily be integrated.

Empty solutions



where $\xi = \sqrt{|3/\Lambda|}$

de Sitter space

$$a = \exp(t/\xi)$$

where $\xi = \sqrt{3/\Lambda}$

$$\Rightarrow H_0 = \sqrt{\frac{\Lambda}{3}} = \sqrt{\frac{8\pi G\rho_\Lambda}{3}}.$$

Today, interesting for two reasons:

- Asymptotic form for models with $\Lambda > 0$ that expand indefinitely
- Inflationary phase

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Today, interesting for two reasons:

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- Inflationary phase

Models without Λ

Astronomers took only these models seriously before 1998.

$$(\dot{a})^2 = \frac{C}{a} - K$$

Three cases:

- $K = 0$ *Einstein-de Sitter universe*
- $K = 1$
- $K = -1$

Einstein-de Sitter universe, $\Lambda = 0, K = 0$

$$a(t) = a_0 \left(\frac{3\sqrt{C}}{2a_0^{3/2}}(t - t_0) + 1 \right)^{2/3}$$

From this, it follows that

$$H_0 = \frac{\sqrt{C}}{a_0^{3/2}}$$

for (slight) simplification:

$$a(t) = a_0 \left(\frac{3}{2} H_0 (t - t_0) + 1 \right)^{2/3}$$

Einstein-de Sitter universe, $\Lambda = 0, k = 0$

This means $a(t_i) = 0$ at

$$t_i = t_0 - \frac{2}{3} H_0^{-1},$$

in other words: the age of the universe is

$$\tau = \frac{2}{3} {H_0}^{-1}.$$

Choose new time coordinate $t - t_i$ and rewrite:

$$a(t) = a_0 \left(\frac{3}{2} H_0 t \right)^{2/3}.$$

Models without Λ, but $K = \pm 1$

Introduce new variable: $u(t) = \sqrt{a(t)/C}$. In this way, one can solve

$$K = +1: \quad t = C\left[\ \sin^{-1}\left(\sqrt{a/C}\right) - \sqrt{a/C - (a/C)^2}\right]$$
$$K = -1: \quad t = C\left[-\sinh^{-1}\left(\sqrt{a/C}\right) + \sqrt{a/C + (a/C)^2}\right]$$

Parametric solution with new parameter χ and

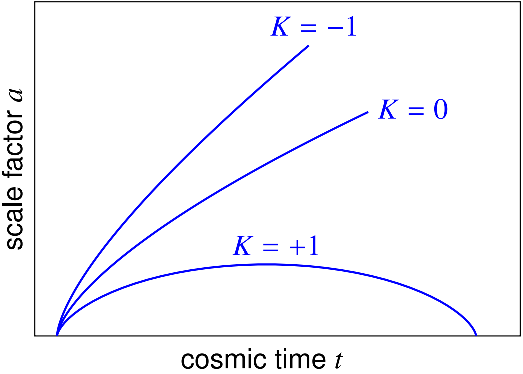
$$a = C \cdot \begin{cases} \sin^2(\chi/2) & \text{for } K = +1 \\ \sinh^2(\chi/2) & \text{for } K = -1 \end{cases}$$

is

$$K = +1: \quad a = \tfrac{1}{2}C \cdot (1 - \cos\chi), \quad t = \tfrac{1}{2}C \cdot (\chi - \sin\chi)$$
$$K = -1: \quad a = \tfrac{1}{2}C \cdot (\cosh\chi - 1), \quad t = \tfrac{1}{2}C \cdot (\sinh\chi - \chi)$$

All models without Λ

Plot the different solutions for $\Lambda = 0$:



Re-collapsing ($K = +1$), borderline ($K = 0$) and expanding ($K = 1$).

The critical density

Evaluate the Friedmann equation

$$3\,\frac{\dot{a}^2 + K}{a^2} - \Lambda = 8\pi G\rho$$

at the present time t_0, absorbing Λ into ρ, to obtain

$$1 = \frac{8\pi G}{3H_0^2}\rho_0 - \frac{K}{a_0 H_0^2}.$$

Where $\rho_0 \equiv \rho(t_0)$. The expression

$$\rho_{c0} \equiv \frac{3H_0^2}{8\pi G}$$

is called the *critical density*.

Critical density and geometry

Using the critical density, rewrite the present-time Friedmann equation as

$$\rho_0/\rho_{c0} = 1 + \frac{K}{a_0 H_0^2}.$$

This equation links the present energy (mass) density ρ_0 of the universe with the Hubble constant H_0 (disguised as ρ_{c0}) and the geometry K:

$\rho_0 > \rho_{c0}$	$\Leftrightarrow$	$K = +1$	spherical space
$\rho_0 = \rho_{c0}$	$\Leftrightarrow$	$K = 0$	Euclidean space
$\rho_0 < \rho_{c0}$	$\Leftrightarrow$	$K = -1$	hyperbolical space

Misconception about critical density and geometry

$\rho_0 > \rho_{c0}$	$\Leftrightarrow$	spherical,	finite,	cosmos will collapse
$\rho_0 = \rho_{c0}$	$\Leftrightarrow$	Euclidean,	infinite,	cosmos will keep expanding
$\rho_0 < \rho_{c0}$	$\Leftrightarrow$	hyperbolical,	infinite,	cosmos will keep expanding

Synonyms: finite = “closed universe”, infinite = “open universe”.

- Local geometry does not control topology!
- Direct correspondence with collapse or not only for $\Lambda = 0$!

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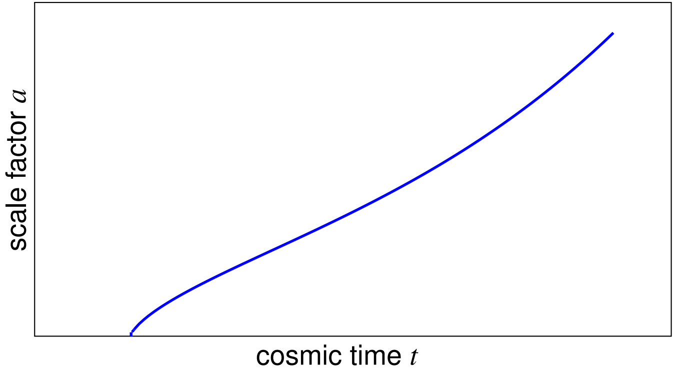
FLRW model with $K = 0, \Lambda > 0$

This is the special case that is probably our own universe:



Explicit solution ⇒ **Exercise**

General considerations for FLRW models

Rescale all present densities in terms of the present critical density, and re-scale K accordingly:

$$\Omega_\Lambda = \rho_\lambda(t_0)/\rho_{c0}, \quad \Omega_M = \rho_M(t_0)/\rho_{c0},$$
$$\Omega_R = \rho_R(t_0)/\rho_{c0}, \quad \Omega_K = -K/(a_0 H_0)^2.$$

Re-write the present-day Friedmann equation as

$$\Omega_\Lambda + \Omega_M + \Omega_R + \Omega_K = 1.$$

This is how densities is linked with spatial geometry.

General considerations for FLRW models

Scaling behaviour of the different densities means that

$$\rho(t) = \frac{3H_0^2}{8\pi G}\left[\Omega_M\left(\frac{a_0}{a(t)}\right)^3 + \Omega_R\left(\frac{a_0}{a(t)}\right)^4 + \Omega_\Lambda\right].$$

Substitute back into the Friedmann equation and substitute $x(t) \equiv a(t)/a_0 = 1/(1+z)$ to obtain

$$\mathrm{d}t = \frac{\mathrm{d}x}{H_0\, x\, \sqrt{\Omega_\Lambda + \Omega_K\, x^{-2} + \Omega_M\, x^{-3} + \Omega_R\, x^{-4}}}.$$

The age of the universe in FLRW models

In the previous models, we defined $t = 0$ by $a(0) = 0$ [initial singularity]. This corresponds to $z \to \infty$ or $x = 0$. With this zero point, emission time $t_E(z)$ of light reaching us with redshift z:

$$t_E(z) = \frac{1}{H_0} \int_0^{1/(1+z)} \frac{\mathrm{d}x}{x \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}.$$

Special case $z = 0$ corresponds to the present time — gives the *age of the universe* τ,

$$\tau = \frac{1}{H_0} \int_0^{1} \frac{\mathrm{d}x}{x \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}.$$

The acceleration parameter q_0

Present pressure:

$$p_0 = \frac{3H_0^2}{8\pi G}(-\Omega_\Lambda + \frac{1}{3}\Omega_R).$$

inserting in

$$q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0),$$

we find that

$$q_0 = \frac{1}{2}(\Omega_M - 2\Omega_\Lambda + 2\Omega_R).$$

The fate of the universe

Rewrite

$$\frac{\dot{a}^2 + K}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G\rho}{3}$$

as

$$\dot{a}^2 = (H_0 a_0)^2 \left[\Omega_\Lambda x^2 + \Omega_M x^{-1} + \Omega_R x^{-2} + \Omega_K\right]$$

where $x = a/a_0$. Forget about Ω_R.

If we want a re-collapse, we must have $\dot{a} = 0$ at some time, in other words:

$$\Omega_\Lambda x^3 + \Omega_M x + \Omega_K x = 0.$$

The fate of the universe

Discussion of

$$\Omega_\Lambda x^3 + \Omega_M x + \Omega_K x = 0$$

- We know that, for $x = 1$, this expression is +1
- For $\Omega_\Lambda < 0$, for sufficiently large u, the expression will become negative $\Rightarrow$ must have a zero
- For $\Omega_\Lambda = 0$, we know recollapse for $\Omega_M > 1$, requiring $K = +1$
- For $\Omega_\Lambda > 0$, recollapse if Ω_K sufficiently negative (again, $K = +1$).

Overview of FLRW solutions

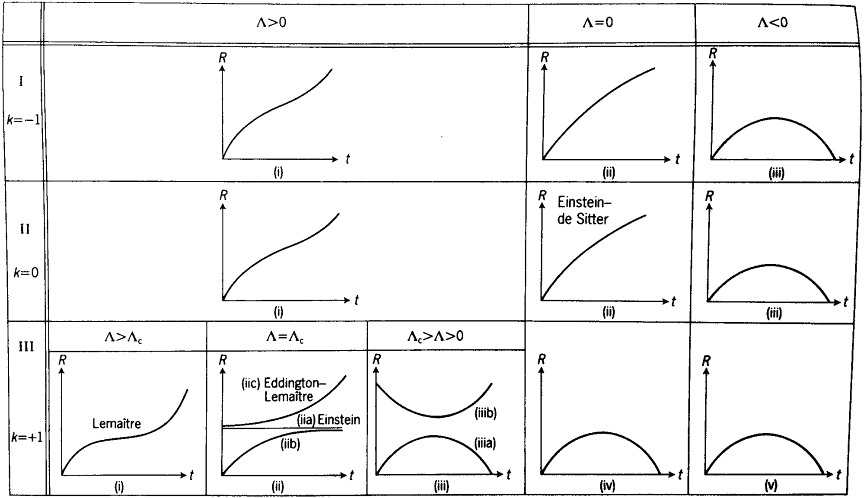


Image from: d'Inverno, *Introducing Einstein's Relativity*, ch. 22.3

Steady State Cosmology

Of historical interest — serious contender 1948 until the 1950s or thereabouts. Invented by Hermann Bondi, Fred Hoyle, Thomas Gold.

Perfect cosmological principle: On average, the universe is isotropic, and *spacetime* is homogeneous (that is, there is no change over time).

Expansion of the de Sitter type:

$$a = \exp(H_0 t)$$

With a continuous creation process of new particles, so that the "steady state" can be maintained.

Local effects of expansion?

Does expansion have an effect locally? Do atoms, planetary orbits, galaxies expand? cf. Giulini, arXiv:1306.0374v1

Overall: Average density means no net force on, say, galaxies ⇒ expansion on largest scales. But what about bound systems?

Pseudo-Newtonian picture: The different inertial frames are "moving away" from each other by the expansion,

$$\ddot{\vec{x}} = \frac{\ddot{a}}{a}\vec{x} \approx -q_0 H_0^2\,\vec{x}$$

gives additional term in Newton's equations,

$$m(\ddot{x} - \frac{\ddot{a}}{a}\vec{x}) = \vec{F}.$$

Only $\ddot{a}$ matters, not $\dot{a}$! Not some "friction force" pulling everything along with expansion!

Expansion and the Coulomb potential 1/3

Setting up a modified Coulomb potential (electromagnetism, gravity): Energy and angular momentum

$$\frac{1}{2}\dot{r}^2 + U(r) = E, \quad r^2\dot{\phi} = L,$$

with the effective potential

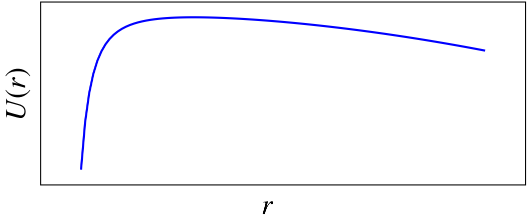
$$U(r) = \frac{L^2}{2r^2} - \frac{C}{r} + \frac{1}{2}Ar^2,$$

where

$$C = \begin{cases} GM & \text{gravitational field} \\ \frac{Qe}{4\pi\epsilon_0 m} & \text{electric field} \end{cases}$$

and $A = -q_0 H_0$.

Expansion and the Coulomb potential 2/3



Critical radius at

$$r_c = \sqrt[3]{\frac{C}{A}}.$$

Amounts to

$$r_c = \begin{cases} \left(\frac{M}{M_\odot}\right)^{1/3} 108\text{pc} & \text{gravity} \\ \left(\frac{Q}{e}\right)^{1/3} 30\text{AU} & \text{electrostatic} \end{cases}$$

Expansion and the Coulomb potential 3/3

$$r_c = \begin{cases} \left(\frac{M}{M_\odot}\right)^{1/3} 108\text{pc} & \text{gravity} \\ \left(\frac{Q}{e}\right)^{1/3} 30\text{AU} & \text{electrostatic} \end{cases}$$

means that:

- for a hydrogen atom, instead of the Sun, the electron would have to be near Pluto
- for the Sun, planets would need to be far beyond the neighbouring stars
- for a galaxy at $10^{12}\ M_\odot$, next galaxy beyond 1 Mpc

Recall $q_0 = \frac{4\pi G}{3}(\rho_0 + 3p_0)$. — for ordinary Dark Energy, density/pressure are constant. If those evolve, as in some *quintessence* models, there could be a “big rip”!

Horizons

Causal structure of spacetime: Which parts are accessible? Which are inaccessible?

Most prominent example: Black holes with their event horizon — what's behind the horizon cannot communicate with the outside. Two varieties: *particle horizon* and *event horizon*.

Particle horizons

In a universe with finite age, the *observable universe* is finite, as well.

Re-writing the FRW metric once more, using $\mathrm{d}s^2 = 0$ to describe light reaching us at the present time, t_0, from some distance r. Light with $r_{\max}$ has been travelling since the big bang ($t = 0$):

$$\int_0^{t_0} \frac{\mathrm{d}t'}{a(t')} = \int_0^{r_{\max}} \frac{\mathrm{d}r'}{\sqrt{1 - Kr'^2}}.$$

But we do not even need to solve for $r_{\max}$, since what we're really interested in is the proper distance:

$$d_{\text{particle}}(t_0) = a(t_0) \int_0^{r_{\max}} \frac{\mathrm{d}r'}{\sqrt{1 - Kr'^2}} = a(t_0) \int_0^{t_0} \frac{\mathrm{d}t'}{a(t')}.$$

Particle horizons

$$d_{\text{particle}}(t_0) = a(t_0)\int_0^{t_0} \frac{\mathrm{d}t'}{a(t')}$$

for special cases:

$$d_{\text{particle}}(t_0) = a(t_0)\int\limits_0^{t_0} \frac{\mathrm{d}t'}{a(t')}$$

for special cases:

Radiation dominated universe: $a(t) \sim \sqrt{t}$, so $H_0 = 1/(2t_0)$, and

$$d_{\text{particle}}(t_0) = 2t_0 = \frac{1}{H_0}.$$

Matter dominated universe: $a(t) \sim \sqrt{t}$, so $H_0 = 1/(2t_0)$, and

$$d_{\text{particle}}(t_0) = 3t_0 = \frac{2}{H_0}.$$

Event horizons

Which events happening at present will we see? Which not?

Same basic derivation from FRW metric:

$$\int_{t_0}^{t_{max}} \frac{dt'}{a(t')} = \int_0^{r_{max(t_0)}} \frac{dr'}{\sqrt{1-Kr'^2}}.$$

t_{max} is infinite for infinitely expanding universes, finite for re-collapsing ones. Calculating proper distances again eliminates the need to solve for $r_{max(t_0)}$:

$$d_{event}(t_0) = a(t_0)\int_0^{r_{max(t_0)}} \frac{dr'}{\sqrt{1-Kr'^2}} = a(t_0)\int_{t_0}^{t_{max}} \frac{dt'}{a(t')}.$$

Event horizons

Most interesting case: our own universe. For large times (follows from explicit solution in exercise)

$$a(t) \sim \exp(\sqrt{\Omega_\Lambda}\, H_0 t),$$

which is dS with $H = \sqrt{\Omega_\Lambda}\, H_0$. Calculate directly that

$$d_{\text{event}}(t_0) = \frac{1}{H} = \frac{1}{\sqrt{\Omega_\Lambda}\, H_0}.$$

For present values of the Hubble constant, that amounts to 8 billion light-years.

What are the next steps?

- Inventory of observational consequences
- How to fix the parameters, test the models
- Separate treatment for early (radiation-dominated) universe
- So far, everything homogeneous: inhomogeneities!

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