cosmology and gravitational lensing

cosmology lecture (chapter 13)

Markus Pössel + Björn Malte Schäfer

Haus der Astronomie and Centre for Astronomy
Fakultät für Physik und Astronomie, Universität Heidelberg

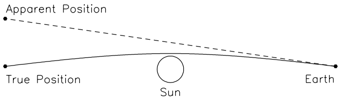
August 6, 2013

outline

repetition

- Friedmann-Lemaître cosmologies with matter and dark energy for accelerated expansion
- thermal history of the universe explains element synthesis and the microwave background
- inflation needed for solving the flatness and horizon-problems, and provides Gaussian initial fluctuations for structure growth
- formation of the cosmic large-scale structure from inflationary perturbations by gravitational instability
- link between statistics and dynamics: linear structure formation is homogeneous (growth equation $D_+(a)$) and conserves Gaussianity of the initial conditions
- halo formation: Jeans-criterion for baryons
- halo density and merging activity determined by Press-Schechter formalism

gravitational lensing: overview



- gravitational light deflection: test of general relativity (1919)
- strong lensing: giant luminous arcs in clusters of galaxies
- weak lensing: correlated distortion of background galaxy images
- multiply imaged quasars and time delays
- lensed light curves of bulge stars and search of MACHOs
- lensing of the microwave background (2007)
- lensing of the microwave background polarisation (2013/2014)

lensing on a point mass

- gravitational fields Φ influence the propagation of light: **Shapiro delay**

$$\Delta t = \int \mathrm{d}x\, \frac{2}{c^3}\Phi \tag{1}$$

light travels slower in a gravitational potential

- we can assign an **index of refraction** to a potential

$$n = 1 - \frac{2}{c^2}\Phi \tag{2}$$

so that the effective speed is $c/n = c - 2\Phi/c$

- we expect lensing effects on gravitational fields due to **Fermat's principle**

$$\hat{\alpha} = -\int \mathrm{d}x\, \nabla_\perp n = \frac{2}{c^2}\int \mathrm{d}x\, \nabla_\perp \Phi \tag{3}$$

lensing on a point mass

- example: gravitational field of a point mass M at distance b, z

$$\Phi(b, z) = -\frac{GM}{\sqrt{b^2 + z^2}} \tag{4}$$

- gradient of the potential

$$\nabla_\perp = \frac{GM}{(b^2 + z^2)^{3/2}}\mathbf{b} \tag{5}$$

where $\mathbf{b}$ points towards the mass and is perpendicular to the ray

- deflection angle:

$$\hat{\alpha} = \frac{2}{c^2}\int \mathrm{d}z\, \nabla_\perp\Phi = \frac{4GM}{c^2 b} \tag{6}$$

weak perturbations of the metric

- consider Minkowski-line element, weakly perturbed by static gravitational potential Φ

$$(\mathrm{d}s)^2 = \left(1 + \frac{2}{c^2}\Phi\right)c^2\mathrm{d}t^2 - \left(1 - \frac{2}{c^2}\Phi\right)\mathrm{d}\vec{x}^2 \qquad (7)$$

- on a geodesic, the line element vanishes: derive effective index of refraction n

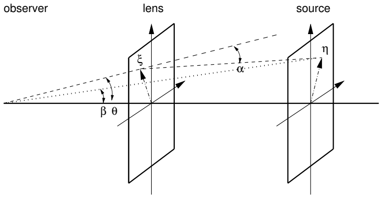
$$\frac{\mathrm{d}|\vec{x}|}{\mathrm{d}t} = c' = \frac{c}{n} \text{ with } n = 1 - \frac{2}{c^2}\Phi \qquad (8)$$

- Fermat's principle: photon minimises run time $\int \left|\mathrm{d}\vec{x}\right| n$

$$\delta \int_{x_i}^{x_f} \mathrm{d}s \sqrt{\frac{\mathrm{d}\vec{x}^2}{\mathrm{d}s^2}} n(\vec{x}(s)) = 0, \qquad (9)$$

for parametrisation $x(s)$ of trajectory with $\left|\mathrm{d}\vec{x}/\mathrm{d}s\right| = 1$

lens equation



- carry out the variation yields ($\nabla_\perp = \nabla - \vec{e}(\vec{e}\nabla)$):

$$\nabla n - \vec{e}(\vec{e}\nabla n) - n\frac{\mathrm{d}\vec{e}}{\mathrm{d}s} = 0 \rightarrow \frac{\mathrm{d}\vec{e}}{\mathrm{d}s} = \nabla_\perp \ln n \simeq -\frac{2}{c^2}\nabla_\perp\Phi \qquad (10)$$

- deflection $\hat{\alpha} = \vec{e}_f - \vec{e}_i = -\frac{2}{c^2}\int \mathrm{d}s \nabla_\perp \Phi$
- read off lens equation, use deflection angle $\hat{\alpha}$:

$$\vec{\eta} = \frac{D_s}{D_l}\vec{\xi} - D_{ls}\hat{\alpha} \rightarrow \beta = \theta - \frac{D_{ls}}{D_s}\hat{\alpha}(\theta) = \theta - \vec{\alpha} \qquad (11)$$

approximations

- formally: $\hat{\alpha} = \vec{e}_f - \vec{e}_i = -\frac{2}{c^2}\int \mathrm{d}s\nabla_\perp\Phi$
- nonlinear integral: the deflection determines the path on which one needs to carry out the integration
- **Born-approximation**: integration along a fiducial straight ray instead of actual photon geodesic
- if the travel path (of order c/H_0)) is large compared to the size of the lens, then the gravitational interaction can be taken to be instantaneous → **thin-lens approximation**
- in this case: project the surface mass density Σ

$$\Sigma(\vec{b}) = \int \mathrm{d}z\,\rho(\vec{b}, z) \tag{12}$$

- deflection is the superposition of all surface density elements

$$\hat{\alpha}(\vec{b}) = \frac{4G}{c^2}\int \mathrm{d}^2b'\,\Sigma(\vec{b}')\frac{\vec{b}-\vec{b}'}{|\vec{b}-\vec{b}'|^2} \tag{13}$$

Einstein radius of a gravitational lens

- Einstein ring: look at deflection

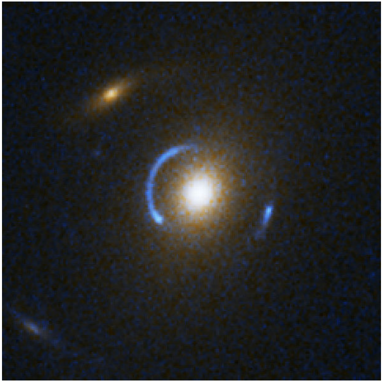
$$\beta = \theta - \alpha = \theta - \frac{D_{ds}}{D_d D_s}\frac{4GM}{c^2\theta} \tag{14}$$

- if the source lies on the optical axis ($\beta = 0$) and if the lens is massive enough al light rays are focused
- we can compute the radius of the ring (in angular units)

$$\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s}} \tag{15}$$

which is called the **Einstein**-radius

strong lensing and Einstein-rings



Einstein ring around an elliptical galaxy, source: SLACS survey

- perfect alignment of source and lens give rise to **Einstein rings**

lens mapping and the mapping Jacobian

- lens equation $\beta = \theta - \vec{\alpha}(\theta)$ relates true position θ to observed position β with mapping field α
- if mapping $\alpha = \nabla_\perp \psi$ is *not constant* across galaxy image $\rightarrow$ distorsion of observed shape
- describe with Jacobian-matrix J

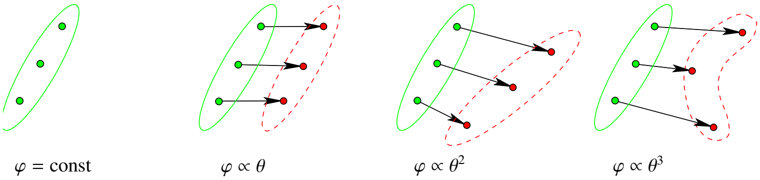
$$J = \frac{\partial \beta}{\partial \theta} = \left(\delta_{ij} - \frac{\partial^2 \psi(\theta)}{\partial \theta_i \partial \theta_j}\right) \tag{16}$$

- decompose $A = \mathrm{id} - J$ in terms of Pauli-matrices:

$$A = \sum_\alpha a_\alpha \sigma_\alpha = \kappa \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \gamma_\times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{17}$$

- coefficients: κ (convergence), γ_+ and $\gamma_\times$ (shear)
- combine shear coefficients to complex shear $\gamma = \gamma_+ + \mathrm{i}\gamma_\times$ (spin 2)

image distortions



- deflection not observable, actual position of a galaxy is unknown
- with assumptions on galaxy ellipticity, the shearing is observable
- bending of an image (flexion) is a new lensing method

question

why is there no rotation of a galaxy image in lensing?

analytical profiles: singular isothermal sphere

- from the part about the stability of self-gravitating systems we know the **singular isothermal sphere**:

$$\rho(r) = \frac{\sigma_\upsilon^2}{2\pi G} \times \frac{1}{r^2} \tag{18}$$

  where the unordered particle motion is described by the velocity disperson σ_υ^2
- compute surface mass density by projection

$$\Sigma(x) = \frac{\sigma_\upsilon^2}{2G} \times \frac{1}{x} \tag{19}$$

- from which we get the deflection angle

$$\hat{\alpha} = 4\pi\frac{\sigma_\upsilon^2}{c^2} \tag{20}$$

mass reconstructions

- convergence $\propto$ local surface mass density Σ of a lens
- but: it is **not directly observable** $\rightarrow$ is it possible to infer κ and the mass map from the observation of gravitational shear?
- write down derivative relations in Fourier space

$$\kappa = -\frac{1}{2}(k_x^2 + k_y^2)\psi \quad \gamma_+ = -\frac{1}{2}(k_x^2 - k_y^2)\psi \quad \gamma_\times = -k_x k_y \psi \tag{21}$$

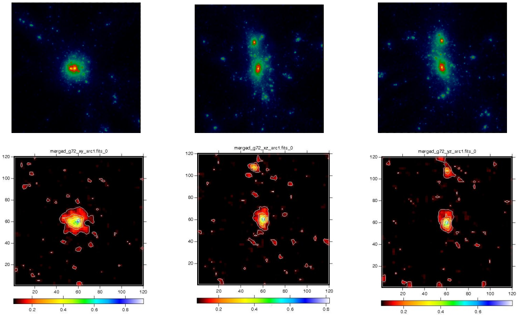
- combine into single equation

$$\left(\begin{array}{c} \gamma_+ \\ \gamma_\times \end{array}\right) = \frac{1}{k^2}\left(\begin{array}{c} k_x^2 - x_y^2 \\ 2k_x k_y \end{array}\right)\kappa \tag{22}$$

- operator is **orthogonal**: $A^2 = \mathrm{id}$

$$\left[\frac{1}{k^2}\left(\begin{array}{c} k_x^2 - k_y^2 \\ 2k_x k_y \end{array}\right)\right]^2 = 1 \tag{23}$$

example: cluster profiles



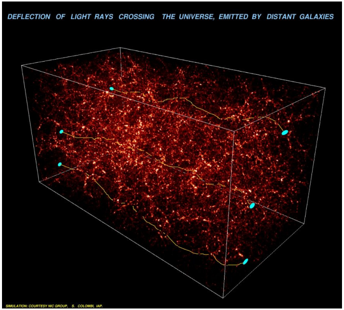
numerical cluster reconstructions, source: J. Merten

- inversion $\kappa = \frac{1}{k^2}\left[(k_x^2 - k_y^2)\gamma_+ + 2k_xk_y\gamma_\times\right]$ yields estimate of map Σ

question

derive the reconstruction operator in real space and formulate the inversion as an integration, identify the Green-function

weak cosmic shear



source: S. Colombi

- lensing on the large-scale structure: fluctuation statistics of the lensing signal reflects the fluctuation statistics of the density field
- neighboring galaxies have correlated deformations because the light rays cross similar, correlated tidal fields

tidal fields and their effect on light rays

- distance x of a gravitationally deflected light ray relative to a fiducial straight line is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\chi^2} = -\frac{2}{c^2}\nabla_\perp\Phi \tag{24}$$

- solution (flat universes)

$$x = \chi\theta - \frac{2}{c^2}\int \mathrm{d}\chi'\,(\chi-\chi')\nabla_\perp\Phi(\chi'\theta) \tag{25}$$

- deflection angle

$$\alpha = \frac{\chi\theta - x}{\chi} = \frac{2}{c^2}\int \mathrm{d}\chi'\,\frac{\chi-\chi'}{\chi}\nabla_\perp\Phi(\chi'\theta) \tag{26}$$

- convergence, with $\nabla_\theta = \chi\nabla_x$

$$\kappa = \frac{1}{2}\mathrm{div}\alpha = \frac{1}{c^2}\int \mathrm{d}\chi'\,(\chi-\chi')\frac{\chi'}{\chi}\Delta\Phi(\chi'\theta) \tag{27}$$

tidal fields and their effect on light rays

- relate to density field with (comoving) Poisson-equation

$$\Delta\Phi = \frac{3H_0^2\Omega_m}{2a}\delta \tag{28}$$

- final result:

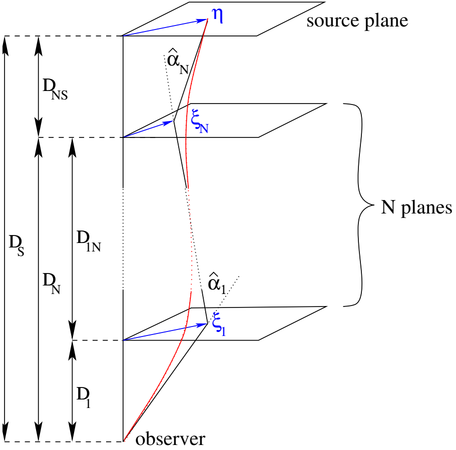
$$\kappa = \int \mathrm{d}\chi' \, W(\chi,\chi')\delta \quad \text{with} \quad W(\chi,\chi') = \frac{3}{2}\left(\frac{H_0}{c}\right)^2 \frac{\Omega_m}{a}(\chi-\chi')\frac{\chi'}{\chi} \tag{29}$$

- fluctuations in κ reflect fluctuations in δ **in a linear way**

cosmic shear

gravitational shear of a galaxy measures the integrated matter density along the line of sight, weighted by $W(\chi)$

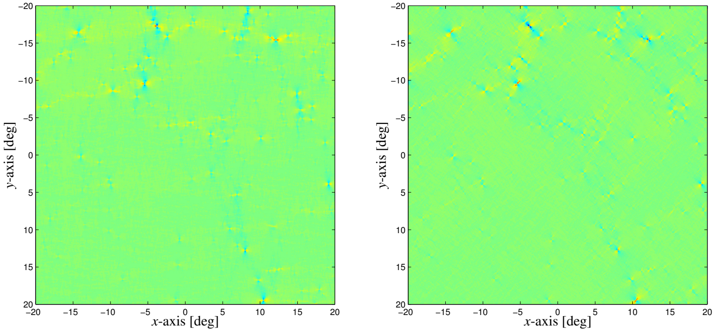
ray-tracing simulations of weak lensing



source: C. Pfrommer

- solve transport $\frac{d^2}{dw^2}x = -\frac{2}{c^2}\nabla_\perp\Phi$ by discretisation

simulated shear field on an n-body simulation



shear γ_+ **shear $\gamma_\times$**

- Gadget-simulated, side length $100\ \mathrm{Mpc}/h$, 40 planes
- clusters of galaxies produce characteristic pattern in shear field

Limber-equation

- original title: Limber (1953), *The Analysis of Counts of the Extragalactic Nebulae in Terms of a Fluctuating Density Field*
- **relate** 3d-power spectrum $P(k)$ to observed 2d-power spectrum $C(\ell)$
- define correlation function $C(\theta) = \langle g(\theta_1) g(\theta_2) \rangle$ of quantity g, which measures fluctuations in density field $g(\theta) = \int \mathrm{d}\chi W(\chi) \delta(\chi\theta, \chi)$
- assume that weighting function $q(\chi)$ does *not* vary much compared to fluctuation scale:

$$C(\theta) = \int \mathrm{d}\chi\, W(\chi)^2 \int \mathrm{d}(\Delta\chi)\, \xi\left(\sqrt{(\chi\theta)^2 + \Delta^2\chi}, \chi\right) \tag{30}$$

- correlation function $C(\theta)$ can be Fourier-transformed to yield angular power spectrum $C(\ell)$:

$$C(\ell) = \int \mathrm{d}\chi \frac{W(\chi)^2}{\chi^2} P\left(k = \frac{\ell}{\chi}, \chi\right) \tag{31}$$

angular spectra

- the spectrum $P(k)$ is defined as

$$\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle = (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')P(k) \quad (32)$$

from the Fourier-transform of the density field $\delta(\mathbf{x})$

$$\delta(\mathbf{k}) = \int \mathrm{d}^3x\,\delta(\mathbf{x})\exp(-\mathrm{i}\mathbf{kx}) \quad\leftrightarrow\quad \delta(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,\delta(\mathbf{k})\exp(+\mathrm{i}\mathbf{kx}) \quad (33)$$

- if the field is not defined in Cartesian coordinates but exists on the surface of the sphere (like an observation at a position on the sky), one needs to use spherical harmonics for decomposition:

$$\gamma(\hat{\theta}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\gamma_{\ell m}Y_{\ell m}(\hat{\theta}) \quad\leftrightarrow\quad \gamma_{\ell m} = \int_{4\pi}\mathrm{d}\Omega\,\gamma(\hat{\theta})Y^*_{\ell m}(\hat{\theta}) \quad (34)$$

and the spectrum reads:

$$\langle\gamma_{\ell m}\gamma^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'}\,C(\ell) \quad (35)$$

Limber-equation: correlation functions

- observable: shear γ at position $\hat{\theta}$ on the sky:

$$\gamma(\hat{\theta}) = \int_0^{\chi_H} \mathrm{d}\chi\, W_\gamma(\chi)\delta(\chi\hat{\theta},\chi) \tag{36}$$

- write down correlation function as the Fourier-transfrom of $P(k)$ and project:

$$C_{\gamma\gamma}(\alpha) = \int_0^{\chi_H} \mathrm{d}\chi W_\gamma(\chi) \int_0^{\chi_H} \mathrm{d}\chi' W_\gamma(\chi') \int \mathrm{d}k k^2 P(k,\chi,\chi') \int_{4\pi} \mathrm{d}\Omega_k \exp(\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}' \tag{37}$$

- correlation function as the Fourier-transform of the spectrum

$$\langle\gamma(\hat{\theta}\chi,\chi)\gamma^*(\hat{\theta}'\chi',\chi')\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} P(k)\exp(\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')) \tag{38}$$

- with the integration done in spherical coordinates

$$\langle\gamma(\hat{\theta}\chi,\chi)\gamma^*(\hat{\theta}'\chi',\chi')\rangle = \int \mathrm{d}k\, k^2 P(k) \int_{4\pi} \mathrm{d}\Omega_k\, \exp(\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')) \tag{39}$$

Limber-equation: Rayleigh-decomposition

- Rayleigh: decomposition of plane waves in spherical waves

$$\exp(\mathrm{i}\mathbf{k}\mathbf{x}) = 4\pi \sum_{\ell=0}^{\infty} \mathrm{i}^{\ell} j_\ell(kx) \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\hat{k}) Y^{*}_{\ell m}(\hat{\theta}) \tag{40}$$

- rewrite Fourier-waves as spherical waves:

$$\int_{4\pi} \mathrm{d}\Omega_k \exp(\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')) = (4\pi)^2 \sum_{\ell=0}^{\infty} j_\ell(k\chi) j_\ell(k\chi') \sum_{m=-\ell}^{+\ell} Y_{\ell m}(\hat{\theta}) Y^{*}_{\ell m}(\hat{\theta}') \tag{41}$$

- use addition theorem of spherical harmonics

$$\int_{4\pi} \mathrm{d}\Omega_k \exp(\mathrm{i}\mathbf{k}(\mathbf{x}-\mathbf{x}')) = 4\pi \sum_{\ell=0}^{\infty} j_\ell(k\chi) j_\ell(k\chi') \, (2\ell+1) \, P_\ell(\cos\alpha) \tag{42}$$

- write correlation function $C_{\gamma\gamma}(\alpha)$ from $P(k)$

$$C_{\gamma\gamma}(\alpha) = 4\pi \int_0^{\chi_H} \mathrm{d}\chi W_\gamma(\chi) \int_0^{\chi_H} \mathrm{d}\chi' W_\gamma(\chi') \int \mathrm{d}k k^2 P(k,\chi,\chi') \sum_{\ell=0}^{\infty} j_\ell(k\chi) j_\ell(k\chi')(\tag{43}$$

Limber-equation: angular spectra

- transform correlation function to ℓ-space by Fourier-transform

$$C_{\gamma\gamma}(\ell) = (4\pi)^2 \int_0^{\chi_H} \mathrm{d}\chi W_\gamma(\chi) \int_0^{\chi_H} \mathrm{d}\chi' W_\gamma(\chi') \int \mathrm{d}k k^2 P(k,\chi,\chi') j_\ell(k\chi) j_\ell(k\chi') \tag{44}$$

- use orthonormality of spherical Bessel functions

$$\int_0^\infty k^2 \mathrm{d}k\, j_\ell(k\chi) j_\ell(k\chi') = \frac{\pi}{2\chi^2} \delta_D(\chi - \chi') \tag{45}$$

- Bessel-functions sort out $P(k) \simeq P(\ell/\chi)$, such that:

$$C_{\gamma\gamma}(\ell) \simeq \int_0^{\chi_H} \frac{\mathrm{d}\chi}{\chi^2} W_\gamma^2(\chi) P(k = \ell/\chi, \chi) \tag{46}$$

Limber-equation

relates fluctuation statistics of the 3d-source field to the statistics of the 2d projected observable

Limber-equation: additional formulas

- angular spectrum from the correlation function

$$C_{\gamma\gamma}(\ell) = 2\pi \int \mathrm{d}\cos\alpha \; C_{\gamma\gamma}(\alpha) P_\ell(\cos\alpha) \tag{47}$$

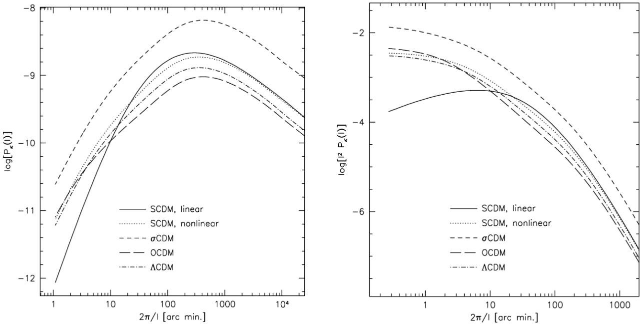
- correlation function from the angular spectrum

$$C_{\gamma\gamma}(\alpha) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) C_{\gamma\gamma}(\ell) P_\ell(\cos\alpha) \tag{48}$$

- addition theorem of the spherical harmonics

$$\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\hat{\theta}) Y^*_{\ell m}(\hat{\theta}') = \frac{2\ell+1}{4\pi} P_\ell(\cos\alpha) \tag{49}$$

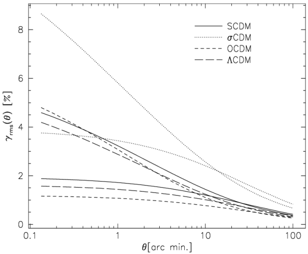
shear power spectra



source: Bartelmann & Schneider, physics reports 340 (2001)

- use Limber's equation to link the shear power spectrum to the dark matter power spectrum
- cosmology: redshift weightings $W(\chi)$, growth $D_+(a(\chi))$, normalisation reflects σ_8

shear in apertures

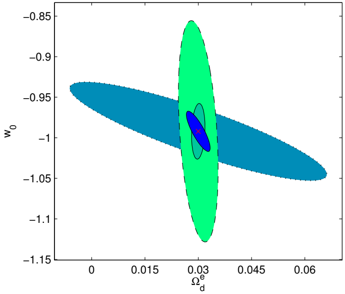


source: Bartelmann & Schneider, physics reports 340 (2001)

- improve constraint on σ_8: $C(\ell)$ should be determined by a small range of k-modes
- average γ in an aperture of size θ: $\langle|\gamma|^2\rangle(\theta)$: product in ℓ-space

$$\langle|\gamma|^2\rangle(\theta) = 2\pi \int_0^\infty \ell \mathrm{d}\ell \, C_\gamma(\ell) \left[\frac{J_1(\theta\ell)}{\pi\theta\ell}\right]^2 \tag{50}$$

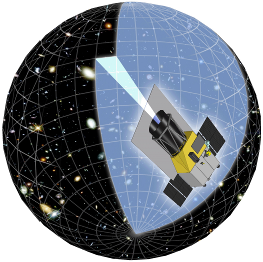
parameter estimates from weak cosmic shear



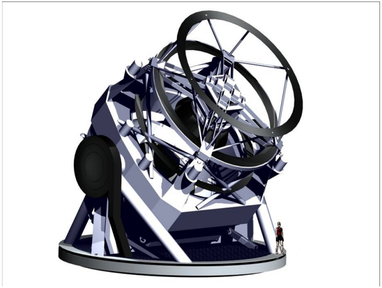
joint constraint on Ω_{EDE} and w_0, source: L. Hollenstein

- lensing is a powerful method for determining parameters
- even complicated dark energy models can be investigated

future lensing surveys



EUCLID



LSST

- coverage $\sim$ half of the sky, going to unit redshift
- precision determination of cosmological parameters, statistical errors $\sim 10^{-3\ldots-4}$
- challenge: **systematics control**

weak lensing tomography

measurements of galaxy shapes

- observe distorsion in the shape of lensed galaxies
- measure second moments of brightness distribution

$$Q_{ij} = \frac{\int \mathrm{d}^2\theta I(\theta)(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)}{\int \mathrm{d}^2\theta I(\theta)} \tag{51}$$

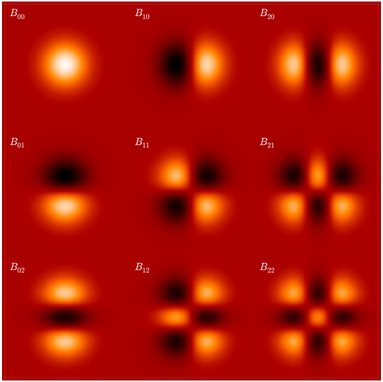
- define complex ellipticity (spin 2):

$$\epsilon = \frac{Q_{xx} - Q_{yy} + 2\mathrm{i}Q_{xy}}{Q_{xx} + Q_{yy} + 2\sqrt{Q_{xx}Q_{yy} - Q_{xy}^2}} \tag{52}$$

- mapping of complex ellipticity by a Jacobian with **reduced shear** $g(\theta) = \gamma(\theta)/[1-\kappa(\theta)]$:

$$\epsilon = \frac{\epsilon' + g}{1 + g^*\epsilon'} \text{ for } |g| \leq 1, \; \epsilon = \frac{1 + (\epsilon')^* g}{(\epsilon')^* - g'} \text{ for } |g| > 1 \tag{53}$$

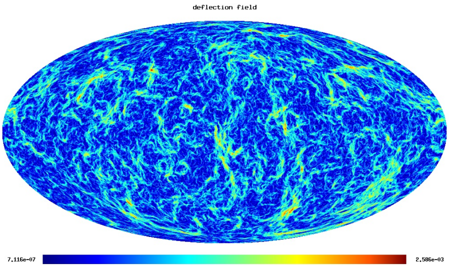
galaxy shapes with shapelets



shapelet base functions B_{ij}, source: P. Melchior

- decomposition into a set of basis functions based on the quantum mechanical harmonic oscillator: Hermite polynomials

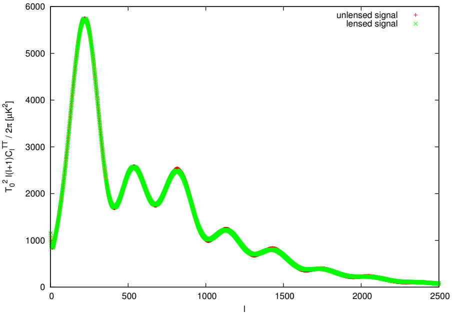
lensing of the cosmic microwave background



sky-map of the deflection angle, source: C. Carbone

- weird (non-Gaussian) patterns in the deflection field
- measurement of lensing at high redshift, in temperature and polarisation

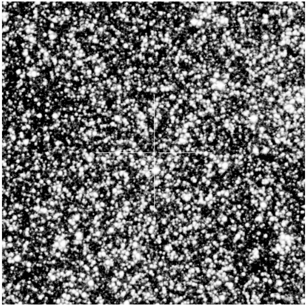
parameter estimates from CMB lensing

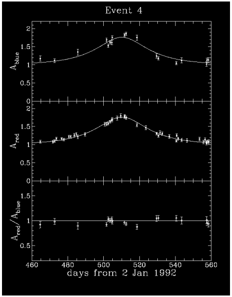


lensed and unlensed CMB spectra, source: Ph. Merkel

- lensing wipes out structures in the CMB (compare to frosted glass)
- amplitudes of the CMB spectrum decreases, non-Gaussianitites in the CMB are generated
- polarisation correlations more strongly affected, B-modes

microlensing and MACHOs

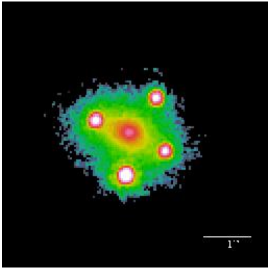


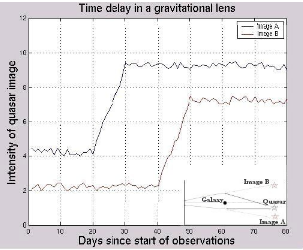


source: C. Alcock

- compact massive objects (historical dark matter candidates) orbit the Milky Way
- observe a large number of bulge stars or stars in the LMC
- find lensed light curves, very typical signature

time delay measurements with quasars





source: universe review

- image appears if the variation of the gravitational time delay is zero
- time delays between different images differ by days
- geometry of the lens can be determined, including the distance

time delay function

- the deflection can be written as the gradient of the lensing potential

$$\theta - \beta = \nabla\psi \tag{54}$$

- which can be combined into a single condition

$$\nabla\left(\frac{1}{2}(\theta-\beta)^2 - \psi\right) = 0 \tag{55}$$

- compare with **time-delay function**

$$\Delta t(\theta) = \frac{1+z}{c}\frac{D_d D_s}{D_{ds}}\left(\frac{1}{2}(\theta-\beta)^2 - \psi\right) = \Delta t_\text{geo} + \Delta t_\text{grav} \tag{56}$$

- the first term corresponds to the time delay along the lensed trajectory, the second term is the Shapiro delay in a gravitational potential
- Fermat's principle now requires $\nabla\Delta t(\theta) = 0$, which might have multiple solutions

summary: Friedmann-Lemaître cosmologies

- **dynamic** world models based on general relativity
- Robertson-Walker line element as a solution to the field equation
- Copernican principle: homogeneous and isotropic metric
- homogeneous fluids, with a certain pressure density relation, parameterised by $w = p/\rho$
  - radiation ($w = +1/3$)
  - (dark) matter ($w = 0$)
  - curvature ($w = -1/3$)
  - cosmological constant ($w = -1$)
- Hubble parameter H_0 defines the critical density $\rho_{\mathrm{crit}} = 3H_0^2/(8\pi G)$
- distance definitions become ambiguous
- geometrical probes constrain the model parameters to a few percent, in particular $\Omega_k < 0.01$

summary: random fields and spectra

- inflation: epoch of rapid **accelerated** expansion of the early universe
- Hubble expansion dominated by a fluid with very negative w
  - drives curvature towards zero → flatness problem
  - grows observable universe from a small volume → horizon problem
- fluctuations in the energy density of the inflaton field couple gravitationally to the other fluids
- fluctuations are Gaussian and have a finite correlation length
  - characterisation with a correlation function $\xi(r)$
  - homogeneous fluctuations: spectrum $P(k)$
- inflationary fluctuations can be observed as temperature anisotropies in the CMB
- shape of the spectrum: inflation gives $P(k) \propto k^{n_s}$, changed by transfer function $T(k)$ in the Meszaros effect, normalised by σ_8

summary: structure formation

- cosmic structures and the large-scale distribution of galaxies form by **gravitational instability** of inflationary perturbation
  - continuity equation
  - Euler equation
  - Poisson equation
- linearisation for small amplitudes: homogeneous growth, described by $D_+(a)$, conservation of Gaussianity of initial conditions
- nonlinear growth is inhomogeneous and destroys Gaussianity by mode coupling
- three basic difficulties
  - nonlinearities in the continuity and Euler-equation
  - collisionlessness of dark matter
  - non-extensivity of gravity
- galaxy formation: gravitational collapse, Jeans argument
- halo density: predicted from $P(k)$ with Press-Schechter formalism

summary: standard model ΛCDM

- ΛCDM is a flat, accelerating Friedmann-Lemaître cosmology with dark matter and a cosmological constant
- ΛCDM has 7 parameters, and is in remarkable agreement with observations, both of geometrical and growth probes
  1. $\Omega_m = 0.25$, low density, required by supernova observations
  2. $\Omega_b = 0.04$, small value, good measurement from CMB
  3. $\Omega_\Lambda = 0.75$, flatness from CMB, $\Omega_m + \Omega_\Lambda = 1$
  4. $w = -1$, cosmological constant, no dynamic dark energy
  5. $\sigma_8 = 0.8$, low value (compared to history), largest uncertainty
  6. $n_s = 0.96$, predicted by inflation to be $\lesssim 1$
  7. $h = 0.72$, sets expansion time scale, or age/size of the universe
- up to now, there is no theoretical understanding of Λ or of the magnitude of H_0

summary: open questions in cosmology

- precision determination of cosmological parameters and verification of the standard model
- matter content of the Universe: dark matter particles, cosmological neutrinos
- inflation, conditions for inflation and observables, Gaussianity
- gravitational waves in the early universe
- quantification of the nonlinearly evolved cosmic density field, description of nonlinear structure formation processes
- substructure of dark matter haloes and an explanation of their kinematic structure
- biasing of galaxies and relations between host halo properties and member galaxies, galaxy formation and evolution
- distinguishing between cosmological constant Λ, dark energy or modified gravity
- tidal interactions of haloes with the large-scale structure